Solution for Stats 369, HW2

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Exercise 1.6.

(a)

For any matrix M by change of variables we get

$$f_M(1) = \frac{1}{2}\int_{\mathsf{S}^{n-1}} \frac{(\det(nM))^{1/2}}{(2\pi)^{n/2}} \exp\left(-\frac{n\sigma^\top M\sigma}{2}\right)\nu_0(\mathrm{d}\sigma)\,,$$

then we have for $z > \lambda_1(W)$,

$$\begin{aligned}\frac{f_{\beta(zI-W)}(1)}{f_I(1)} &= \det\left(\beta(zI-W)\right)^{1/2}\cdot\frac{\int_{\mathsf{S}^{n-1}}\exp\left(-\frac{n\beta\sigma^\top(zI-W)\sigma}{2}\right)\nu_0(\mathrm{d}\sigma)}{\int_{\mathsf{S}^{n-1}}\exp\left(-\frac{n\sigma^\top\sigma}{2}\right)\nu_0(\mathrm{d}\sigma)}\\ &= \beta^{n/2}\cdot\det\left(zI-W\right)^{1/2}\cdot\frac{\exp\left(-\frac{nz\beta}{2}\right)\cdot\int_{\mathsf{S}^{n-1}}\exp\left(\frac{n\beta\sigma^\top W\sigma}{2}\right)\nu_0(\mathrm{d}\sigma)}{\exp\left(-\frac{n}{2}\right)}\\ &= \beta^{n/2}\cdot\exp\left(-\frac{n(z\beta-1)}{2}\right)\cdot\det\left(zI-W\right)^{1/2}\cdot Z_n(\beta;W)\,.\end{aligned}$$

Rearranging terms concludes the proof of part (a).

(b)

By the explicit formula for $Z_n(\beta;W)$ in (1.7.13) we can write

$$\frac{1}{n}\log Z_n(\beta;W) = -\frac{1}{2}\log\beta + \frac{1}{2}(z\beta-1) - \frac{1}{2n}\sum_{i=1}^n \log(z-\lambda_i(W)) + \frac{1}{n}\log\frac{f_{\beta(zI-W)}(1)}{f_I(1)}$$

for any $z > \lambda_1(W)$. Since $\lambda_1(W)\to 2$ the above equation holds eventually for any $z > 2$. Note that by direct calculations

$$\begin{aligned}\int \log(z-\lambda)s_\infty(\mathrm{d}\lambda) &= \frac{z^2}{4} - \frac{1}{2} - \frac{z\sqrt{z^4-4}}{4} + \log\left(\frac{z+\sqrt{z^2-4}}{2}\right)\\ \int\frac{1}{z-\lambda}s_\infty(\mathrm{d}\lambda) &= \frac{z-\sqrt{z^2-4}}{2}\,.\end{aligned}$$

For any $z > 2$ we have $\int\frac{1}{z-\lambda}s_\infty(\mathrm{d}\lambda)$ is increasing in z and takes values in $(0,1)$. The equation

$$\beta = \int\frac{1}{z-\beta}s_\infty(\mathrm{d}\lambda) = \frac{z-\sqrt{z^2-4}}{2}$$

indeed has a unique solution $z = \beta + \frac{1}{\beta} > 2$ for any chosen $\beta < 1$. For this z, since $\lambda_1(W)\to 2$ we have $\log(z-\lambda_i(W))$ will eventually be bounded, and $\log(z-\lambda)$ is a continuous function, we can deduce from weak convergence $s_n \xrightarrow{w} s_\infty$ that

$$\frac{1}{2n}\sum_{i=1}^{n}\log(z-\lambda_i(W)) = \frac{1}{2}\int\log(z-\lambda)s_n(\mathrm{d}\lambda) \to \frac{1}{2}\int\log(z-\lambda)s_\infty(\mathrm{d}\lambda)\,.$$

Now it only remains to show with probability 1 that

$$\frac{1}{n}\log\frac{f_{\beta(zI-W)}(1)}{f_I(1)} \to 0\,.$$

In fact we have $|\log f_{\beta(zI-W)}(1)| = o(n)$ and $|\log f_I(1)| = o(n)$. For $x \sim \mathsf{N}(0, n\beta(zI-W)^{-1})$, we have

$$\|x\|^2 = \frac{1}{n\beta}\sum_{j=1}^{n}\lambda_j^{-1}v_j^2, \qquad v_j \overset{i.i.d.}{\sim} \mathsf{N}(0,1)\,,$$

where we use the shorthand $\lambda_j = \lambda_j(zI-W)$. Let $X_j = \frac{1}{\beta\lambda_j}(v_j^2-1)$, then $\{X_j\}$ are independent random variables with zero mean and

$$\|x\|^2 = \frac{1}{n}\sum_{j=1}^{n}X_j + \frac{1}{n\beta}\sum_{j=1}^{n}\lambda_j^{-1}.$$

Denote by

$$\sigma_n^2 = \sum_{j=1}^{n}\mathbb{E}\left[X_j^2\right] = \frac{2}{\beta^2}\sum_{j=1}^{n}\lambda_j^{-2} = \Theta(n),$$

we can deduce from local central limit theorem that

$$f_{\beta(zI-W)}(1) = \frac{n}{\sigma_n}\cdot\left\{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1-\frac{1}{n\beta}\sum_{j=1}\lambda_j^{-1}}{2\sigma_n^2/n^2}\right) + O\left(\frac{1}{\sigma_n}\right)\right\}.$$

Using the fact that $\frac{1}{n\beta}\sum_{j=1}\lambda_j^{-1} \to \frac{1}{\beta}\int\frac{1}{z-\lambda}s_\infty(\mathrm{d}\lambda) = 1$ yields $f_{\beta(zI-W)}(1) = \exp(o(n))$. Similarly we have $f_I(1) = \exp(o(1))$.

(c)

Since

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log Z_n(\beta;W) &= -\frac{1}{2}\log\beta + \frac{1}{2}\beta z - \frac{1}{2} - \frac{1}{2}\int\log(z-\lambda)s_\infty(\mathrm{d}\lambda)\\ &= -\frac{1}{2}\log\beta + \frac{1}{2}\beta z - \frac{1}{2} - \frac{1}{2}\left\{\frac{z^2}{4} - \frac{1}{2} - \frac{z\sqrt{z^4-4}}{4} + \log\left(\frac{z+\sqrt{z^2-4}}{2}\right)\right\},\end{aligned}$$

we can substitute in $z = \beta + \frac{1}{\beta}$ and obtain

$$\phi(\beta) = -\frac{1}{2}\log\beta + \frac{1}{2}\beta^2 - \frac{1}{2}\left\{\frac{1}{4}\left(\beta^2+\frac{1}{\beta^2}\right) + \frac{1}{4}\left(\beta^2 - \frac{1}{\beta^2}\right) - \log\beta\right\} = \frac{\beta^2}{4}\,.$$

For $k=2$ and $\lambda = 0$ we have

$$\Psi_{\mathrm{RS}}(q,\beta) = \frac{\beta^2}{4}\left(1-q^2\right) + \frac{q}{2(1-q)} + \frac{1}{2}\log(1-q)\,.$$

Note that $\Psi_{\mathrm{RS}}(q,\beta)$ is increasing in $q \in [0,1]$ and thus

$$\phi(\beta) = \inf_{\beta\in[0,1]}\Psi_{\mathrm{RS}}(q,\beta) = \Psi_{\mathrm{RS}}(0,\beta) = \frac{\beta^2}{4}\,.$$

The two solutions coincide.

Exercise 1.7.

We copy here the free energy density for 1RSB,

$$\begin{aligned}\Psi_{\text{1RSB}}(b,q_0,q_1,m) =& \frac{\beta\lambda}{\sqrt{2(k!)}}b^k + \frac{\beta^2}{4}[1-(1-m)q_1^k - mq_0^k] + \frac{1}{2}\frac{q_0-b^2}{1-(1-m)q_1-mq_0} \\ &+ \frac{1}{2m}\log(1-(1-m)q_1-mq_0) - \frac{1-m}{2m}\log(1-q_1).\end{aligned} \tag{1}$$

Taking $\lambda = 0$, $b = 0$, and $q_0 = 0$, we have

$$\Psi_{\text{1RSB}}(q_1,m;\beta) = \frac{\beta^2}{4}[1-(1-m)q_1^k] + \frac{1}{2m}\log(1-(1-m)q_1) - \frac{1-m}{2m}\log(1-q_1). \tag{2}$$

(a)

See the matlab program below.

```
clc
clear
close all

kset = [3,4,5];
for iterk = 1:length(kset)
    k = kset(iterk);

    mset = [0.01:0.01:1.99]';
    m = 1;
    fq0 = @(q1) q1.^(k-2).*(1-q1).*(1-(1-m)*q1);
    q1 = 0.0001:0.0001:0.9999;
    [fstar, qid] = max(fq0(q1));
    Tdm1 = sqrt(fstar*k/2)
    Tdm1value(iterk) = Tdm1;
    Tset = [1.5*Tdm1:-0.01:0.01];
    Tsetvalue{iterk} = Tset;

    mstar = 1;
    for iterT = 1:length(Tset)
        T = Tset(iterT)
        beta = 1/T;
        Psi = @(q1,m) beta^2/4 * (1 - (1-m).*q1.^k)+1./(2*m).*log(1-(1-m).*q1)
            - (1-m)./(2.*m).*log(1-q1);
        for iterm = 1:length(mset)
            m = mset(iterm);
            fq0 = @(q1) q1.^(k-2).*(1-q1).*(1-(1-m)*q1);
            q1 = 0.001:0.001:0.999;
            [fstar, qid] = max(fq0(q1));
            Tdm = sqrt(fstar*k/2);
            Tdmvalue{iterk} = Tdm;

            if beta <= 1/Tdm
                q1starm = 0;
            else
                fq = @(q1) q1.^(k-2).*(1-q1).*(1-(1-m)*q1) - 2/(k*beta^2);
                q1 = 0.001:0.001:0.999;
                qid = max(find((fq(q1) > 0) == 1));
```

```
                q1starm = q1(qid);
            end
            Psistarm(iterm) = Psi(q1starm,m);
        end

        %plot(mset,Psistarm-beta^2/4);
        [Psimin, iterm] = min(Psistarm);
        mstar = mset(iterm);
        if mset(iterm) > 1
            Psimin = beta^2/4;
            Ts = T;
            mstar = 1;
            q1starm = 0;
        end
        if min(Psistarm-beta^2/4) > -1e-6
            mstar = 1;
        end
        Psivalue(iterk,iterT) = Psimin;
        mvalue(iterk,iterT) = mstar;
        q1value(iterk,iterT) = q1starm;
    end
end
save('result.mat');
```

(b)

See Figure 1,2,3.

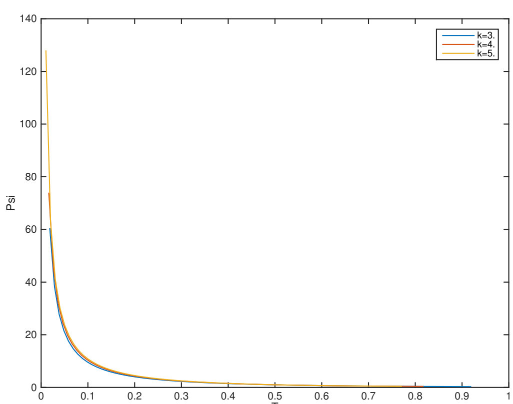

Figure 1: $\psi_{\mathrm{1RSB}}(\mathrm{T})$.

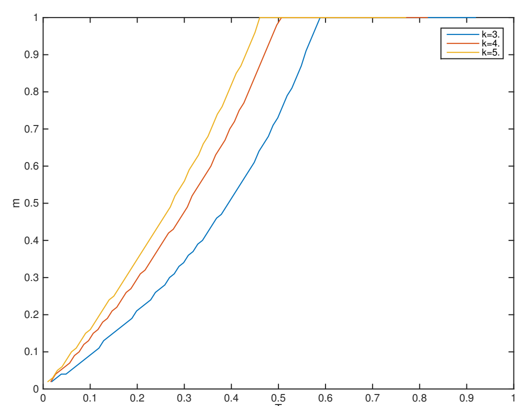

Figure 2: $m(T)$

(c)

k	3	4	5	6	7	8	9	10
T_s	0.5885	0.5064	0.4602	0.4436	0.4162	0.4040	0.3949	0.3879

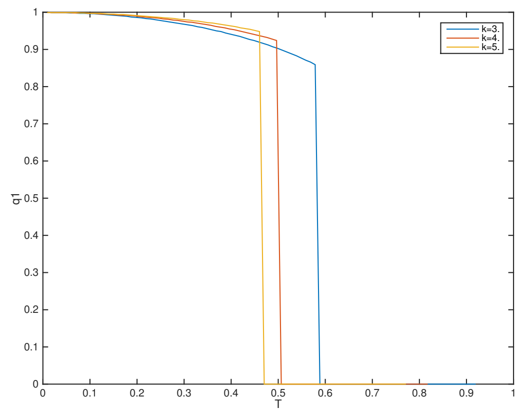

Figure 3: q_1(T).

Exercise 1.8.

(a)

Since we have

$$
\begin{aligned}
&\Psi_{1\mathrm{RSB}}(1-c(\mu)T+o(T),\mu T;T)\\
=&\frac{1}{4T^2}[1-(1-\mu T)(1-c(\mu)T+o(T))^k]+\frac{1}{2\mu T}\log(1-(1-\mu T)(1-c(\mu)T+o(T)))-\frac{1-\mu T}{2\mu T}\log(c(\mu)T+o(T))\\
=&\frac{1}{4T^2}[(kc(\mu)+\mu)T+o(T)]+\frac{1}{2\mu T}\log((\mu+c(\mu))T+o(T))-\frac{1-\mu T}{2\mu T}\log(c(\mu)T+o(T)).
\end{aligned}
\tag{3}
$$

The coefficient of $O(1/T)$ term (which is the leading order term as $T \to 0$) gives

$$
e(\mu)=\frac{1}{4}kc(\mu)+\mu+\frac{1}{2\mu}\log(\mu+c(\mu))-\frac{1}{2\mu}\log(c(\mu)). \tag{4}
$$

We have

$$
\frac{\mathrm{d}}{\mathrm{d}\mu}e(c(\mu))=\frac{1}{4}k+\frac{1}{2\mu}\frac{1}{\mu+c(\mu)}-\frac{1}{2\mu}\frac{1}{c(\mu)}=0, \tag{5}
$$

which gives

$$
c^2+c\mu-\frac{2}{k}=0. \tag{6}
$$

Thus, we have

$$
c(\mu)=\frac{\sqrt{\mu^2+8/k}-\mu}{2}. \tag{7}
$$

(b)

We have

$$
\Psi_{1\mathrm{RSB}}(q_{1,*}(\beta,\mu T),\mu T;\beta)=\beta e_{1\mathrm{RSB}}+o(\beta). \tag{8}
$$

Substituting $c(\mu)$ into the $e(\mu)$, we have

$$
e_{1\mathrm{RSB}}(\mu)=\frac{1}{\sqrt{\mu^2+8/k}+\mu}+\frac{1}{4}\mu+\frac{1}{2\mu}\log(1+\frac{2\mu}{\sqrt{\mu^2+8/k}-\mu}). \tag{9}
$$

(c)

e^*_{1RSB} is the ground state energy of the k-spin glass model. That is,

$$e^*_{\mathrm{1RSB}} = \lim_{n\to\infty} \mathbb{E}[\max_{\sigma\in\mathbb{S}^{n-1}} \frac{1}{\sqrt{2(k!)}}\langle J, \sigma^{\otimes k}\rangle]. \tag{10}$$

The value for e^*_{1RSB} for different k gives

k	2	3	4	5	6	7	8	9	10
e^*_{1RSB}	1.0000	1.1717	1.2686	1.3350	1.3850	1.4248	1.4577	1.4857	1.5099